II. CANONICAL QUANTIZATION OF ELECTRODYNAMICS:

With the foregoing preparation, we are now in a position to apply the *classical analogy* or *canonical quantization* program to achieve the ***second quantization*** of the electromagnetic field.[5] As our starting point and for reference, we, once again, set forth the vacuum or microscopic Maxwell's equations in the time domain:

$$\vec{\nabla} \times \vec{\mathbf{E}}(\vec{\mathbf{r}},t) = -\frac{\partial}{\partial t}\vec{\mathbf{B}}(\vec{\mathbf{r}},t) \qquad \text{[II-1a]}$$

$$\vec{\nabla} \times \vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},t) + \varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{\mathbf{E}}(\vec{\mathbf{r}},t) \qquad \text{[II-1b]}$$

$$\vec{\nabla} \cdot \vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t)/\varepsilon_0 \qquad \text{[II-1c]}$$

$$\vec{\nabla} \cdot \vec{\mathbf{B}}(\vec{\mathbf{r}},t) = 0 \qquad \text{[II-1d]}$$

The canonical formulation of classical electrodynamics (**Jeans' Theorem**) is most conveniently achieved in terms of the (magnetic) vector potential in the time domain -- *viz*.

$$\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \vec{\nabla} \times \vec{\mathbf{A}}(\vec{\mathbf{r}},t) \qquad \text{[II-2a]}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = -\frac{\partial}{\partial t}\vec{\mathbf{A}}(\vec{\mathbf{r}},t) - \vec{\nabla}\phi(\vec{\mathbf{r}},t) \qquad \text{[II-2b]}$$

so that

$$\vec{\nabla}\left[\vec{\nabla} \cdot \vec{\mathbf{A}}(\vec{\mathbf{r}},t)\right] - \nabla^2\vec{\mathbf{A}}(\vec{\mathbf{r}},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\mathbf{A}}(\vec{\mathbf{r}},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\phi(\vec{\mathbf{r}},t) = \mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},t) \qquad \text{[II-3a]}$$

---

5 In common usage, the process of treating the cordinates q_i and p_i as quantized variables is called *first quantization*. *Second quantization* is the process of quantizing ***fields*** -- say, $\vec{\mathbf{A}}(\vec{\mathbf{r}},t)$ -- which have an infinite number of dequees of freedom.

$$-\varepsilon_0 \vec{\nabla} \frac{\partial}{\partial t}\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right) - \varepsilon_0 \nabla^2 \phi\left(\vec{\mathbf{r}},t\right) = \rho\left(\vec{\mathbf{r}},t\right) w \qquad \text{[II-3b]}$$

In QED (Quantum Electrodynamics) it is *convenient and traditional* to make use of the **Coulomb gauge --** *i.e.* $\vec{\nabla}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right) = 0$ -- so that

$$\nabla^2 \vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right) = -\mu_0 \vec{\mathbf{J}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[II-4a]}$$

$$\nabla^2 \phi\left(\vec{\mathbf{r}},t\right) = -\rho\left(\vec{\mathbf{r}},t\right)/\varepsilon_0 \qquad \text{[II-4b]}$$

where $\vec{\mathbf{J}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) = \vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right) - \vec{\mathbf{J}}_{\mathrm{L}}\left(\vec{\mathbf{r}},t\right) = \vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right) - \varepsilon_0 \frac{\partial}{\partial t}\vec{\nabla}\phi\left(\vec{\mathbf{r}},t\right)$ is the so called ***transverse*** current density. Since $\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right)$ is completely determined by the transverse current density in the Coulomb gauge, electromagnetic problems become in a sense separable -- *i.e.*

The *transverse* field problem:

$$\begin{aligned} &\vec{\nabla}\,\vec{\mathbf{E}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) = 0 \\ &\vec{\nabla}\times\vec{\mathbf{E}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) = -\mu_0 \frac{\partial}{\partial t}\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right) \\ &\vec{\nabla}\times\vec{\mathbf{H}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) = \vec{\mathbf{J}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathbf{E}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) \end{aligned} \qquad \text{[II-5a]}$$

The *longitudinal* field problem:

$$\begin{aligned} &\vec{\nabla}\,\vec{\mathbf{E}}_{\mathrm{L}}\left(\vec{\mathbf{r}},t\right) = \rho\left(\vec{\mathbf{r}},t\right)/\varepsilon_0 \\ &\vec{\mathbf{J}}_{\mathrm{L}}\left(\vec{\mathbf{r}},t\right) = -\varepsilon_0 \frac{\partial}{\partial t}\vec{\mathbf{E}}_{\mathrm{L}}\left(\vec{\mathbf{r}},t\right) \end{aligned} \qquad \text{[II-5b]}$$

We turn now explicitly to a treatment of the ***free electromagnetic field*** -- formally the case of $\vec{\mathbf{J}}_T(\vec{\mathbf{r}},t) \equiv 0$ wherein

$$\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = 0 \qquad \text{[II-6]}$$

We look for solutions in the form

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{1}{2\sqrt{\varepsilon_0}}\sum_s \left\{ \alpha_s(t)\,\vec{\mathbf{u}}_s(\vec{\mathbf{r}}) + c.c. \right\} \qquad \text{[II-7]}$$

Substituting into Equation [II-6] we obtain

$$\sum_s \left\{ \alpha_s(t)\,\vec{\mathbf{u}}_s(\vec{\mathbf{r}}) \left[\frac{\nabla^2 \vec{\mathbf{u}}_s(\vec{\mathbf{r}})}{\vec{\mathbf{u}}_s(\vec{\mathbf{r}})} - \frac{1}{c^2}\frac{\ddot{\alpha}_s(t)}{\alpha_s(t)} \right] + c.c. \right\} = 0 \qquad \text{[II-8]}$$

Thus we may apply **separation of variables** techniques and the original problem is divided in to **two new and distinct** problems -- *viz.* solutions of the following set of equations:

$$\nabla^2 \vec{\mathbf{u}}_s(\vec{\mathbf{r}}) + \frac{\omega_s^2}{c^2}\vec{\mathbf{u}}_s(\vec{\mathbf{r}}) = 0 \qquad \text{[II-9]}$$

$$\ddot{\alpha}_s(t) + \omega_s^2\,\alpha_s(t) = 0 \qquad \text{[II-10]}$$

where the ω_s's are **separation constants**. The spatial equations allow us to treat the boundary value problem of the cavity or defining field space in whatever detail that might seem appropriate in a particular case.[6] But the *bottom line* is that the boundary condition

---

6 The $\vec{\mathbf{u}}_i(\vec{\mathbf{r}})$'s do not form a true complete set of solutions since no longitudinal vector field can be expanded in terms of such *divergentless* functions.

taken together with the characteristics of the normal modes or eigenfunctions determine the ω_s's which may, in turn, be identified as the eigenfrequencies of the normal modes.

Thus, we may now write

$$\vec{\mathbf{E}}_T(\vec{\mathbf{r}},t) = -\frac{\partial}{\partial t}\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = -\frac{1}{2\sqrt{\varepsilon_0}}\sum_s\left\{\dot{\alpha}_s(t)\,\vec{\mathbf{u}}_s(\vec{\mathbf{r}}) + c.c.\right\}$$
$$= \frac{1}{2\sqrt{\varepsilon_0}}\sum_s\left\{i\,\omega_s\alpha_s(t)\,\vec{\mathbf{u}}_s(\vec{\mathbf{r}}) + c.c.\right\} \qquad \text{[II-11]}$$

and

$$\vec{\mathbf{H}}_T(\vec{\mathbf{r}},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{c}{2\sqrt{\mu_0}}\sum_s\left\{\alpha_s(t)\,\vec{\nabla}\times\vec{\mathbf{u}}_s(\vec{\mathbf{r}}) + c.c.\right\}. \qquad \text{[II-12]}$$

The crucial step required in establishing Jeans' Theorem is the expansion of the instantaneous value of the stored electromagnetic energy in terms of the cavity modes -- *viz.*

$$\langle W(t)\rangle = \frac{1}{2V}\iiint\left[\varepsilon_0\left|\vec{\mathbf{E}}(\vec{\mathbf{r}},t)\right|^2 + \mu_0\left|\vec{\mathbf{H}}(\vec{\mathbf{r}},t)\right|^2\right]dV \qquad \text{[II-13a]}$$

which, in light of Equations [II-11] and [II-12], becomes

$$\langle W(t)\rangle = \frac{1}{4V}\sum_{s,s'}\iiint\left[\omega_s\omega_{s'}\,\vec{\mathbf{u}}_s\cdot\vec{\mathbf{u}}^*_{s'} + c^2\,\vec{\nabla}\times\vec{\mathbf{u}}_s\cdot\vec{\nabla}\times\vec{\mathbf{u}}^*_{s'}\right]\alpha_s(t)\,\alpha^*_{s'}(t)\,dV \qquad \text{[II-13b]}$$

To proceed we need the value of the integral $\iiint_{\text{cavity}} \left[\vec{\nabla}\times\vec{\mathbf{u}}_{s} \cdot \vec{\nabla}\times\vec{\mathbf{u}}^{*}_{s'}\right] dV$. Using a well known vector identity,[7] we obtain

$$\iiint \left[\vec{\nabla}\times\vec{\mathbf{u}}_{s} \cdot \vec{\nabla}\times\vec{\mathbf{u}}^{*}_{s'}\right] dV = \iint \vec{\mathbf{u}}_{s}\times\left(\vec{\nabla}\times\vec{\mathbf{u}}^{*}_{s'}\right)\cdot d\vec{S} + \iiint \vec{\mathbf{u}}_{s}\cdot\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{u}}^{*}_{s'}\right) dV \qquad \text{[II-14a]}$$

By boundary value arguments we may easily show that the first term on the RHS of this equation vanishes and by using an even more familiar (famous) vector identity[8] for a divergenceles field, we obtain

$$\iiint \left[\vec{\nabla}\times\vec{\mathbf{u}}_{s} \cdot \vec{\nabla}\times\vec{\mathbf{u}}^{*}_{s'}\right] dV = -\iiint \vec{\mathbf{u}}_{s}\cdot\nabla^{2}\vec{\mathbf{u}}^{*}_{s'}\, dV = \frac{\omega^{2}_{s'}}{c^{2}} \iiint \vec{\mathbf{u}}_{s}\cdot\vec{\mathbf{u}}^{*}_{s'}\, dV \qquad \text{[II-14b]}$$

Therefore, Equation [II-13b] becomes

$$\langle W(t)\rangle = \frac{1}{4V}\sum_{s,s'}\left[\omega_{s}\,\omega_{s'} + \omega^{2}_{s'}\right]\alpha_{s}(t)\,\alpha^{*}_{s'}(t)\iiint \vec{\mathbf{u}}_{s}\cdot\vec{\mathbf{u}}^{*}_{s'}\, dV \qquad \text{[II-15]}$$

From Equation [II-14b] we may also write

---

7 Namely, that $\vec{\nabla}\cdot\left(\vec{\mathbf{X}}\times\vec{\nabla}\times\vec{\mathbf{Y}}\right) = \left(\vec{\nabla}\times\vec{\mathbf{X}}\right)\cdot\left(\vec{\nabla}\times\vec{\mathbf{Y}}\right) - \vec{\mathbf{X}}\cdot\left(\vec{\nabla}\times\vec{\nabla}\times\vec{\mathbf{Y}}\right)$

8 Namely, that

$$\begin{aligned}\vec{\nabla}\cdot\left[\vec{\mathbf{X}}\times\left(\vec{\nabla}\times\vec{\mathbf{Y}}\right)\right] &= \left(\vec{\nabla}\times\vec{\mathbf{X}}\right)\cdot\left(\vec{\nabla}\times\vec{\mathbf{Y}}\right) - \vec{\mathbf{X}}\cdot\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{Y}}\right)\\ &= \left(\vec{\nabla}\times\vec{\mathbf{X}}\right)\cdot\left(\vec{\nabla}\times\vec{\mathbf{Y}}\right) - \vec{\mathbf{X}}\cdot\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{Y}}\right) + \vec{\mathbf{X}}\cdot\nabla^{2}\vec{\mathbf{Y}}\end{aligned}$$

$$\iiint \vec{\ } \times \left[\vec{\mathbf{u}}_{s} \times \vec{\ } \times \vec{\mathbf{u}}^{*}_{s'} - \vec{\mathbf{u}}^{*}_{s'} \times \vec{\ } \times \vec{\mathbf{u}}_{s} \right] dV = \frac{1}{c^2}\left(\omega^2_{s'} - \omega^2_{s}\right) \iiint \vec{\mathbf{u}}_{s} \;\; \vec{\mathbf{u}}^{*}_{s'} \; dV \text{ II-16a]}$$

and, again, by boundary value arguments we may easily show that the resultant surface imtegral on the left v

$$\left(\omega^2_{s'} - \omega^2_{s}\right) \iiint \vec{\mathbf{u}}_{s} \;\; \vec{\mathbf{u}}^{*}_{s'} \; dV = 0 \qquad \text{[II-16b]}$$

Therefore, we may take

$$\iiint \left[\vec{\mathbf{u}}_{s} \;\; \vec{\mathbf{u}}^{*}_{s'} \right] dV = \delta_{ss'} \qquad \text{[II-17]}$$

so that Equation [II-15] becomes

$$\langle W(t) \rangle = \frac{1}{2} \sum_{s} \omega^2_{s} \, \alpha_{s} \, \alpha^{*}_{s} \qquad \text{[II-18]}$$

which when compared to Equation [I-6][9] **is effectively the content of Jean's Theorem** with

$$\begin{aligned} &\sqrt{\frac{\omega_s}{2}} \, \alpha_s \qquad a = \sqrt{\frac{m}{2\,\omega}} \left[\omega \, q + i \, p/m\right] \\ &\sqrt{\frac{\omega_s}{2}} \, \alpha^{*}_s \qquad a^{\dagger} = \sqrt{\frac{m}{2\,\omega}} \left[\omega \, q - i \, p/m\right] \end{aligned} \qquad \text{[II-19]}$$

---

9 That is $\mathcal{H} = \omega \, a^{\dagger} \, a$

To accomplish the *canonical quantization* program, field variables are expressed as field operators by making the identification

$$\begin{aligned} \sqrt{\frac{\omega_s}{2}}\,\alpha_s \quad & \sqrt{\frac{m}{2\,\omega}}\left[\omega\, q + i\; p\,/\,m\right] = \sqrt{\hbar}\, a \\ \sqrt{\frac{\omega_s}{2}}\,\alpha^*_s \quad & \sqrt{\frac{m}{2\,\hbar\omega}}\left[\omega\, q - i\; p\,/\,m\right] = \sqrt{\hbar}\, a^{\dagger} \end{aligned} \qquad \text{[II-20]}$$

which leads to the following set of field operators:

$$\mathcal{H} = \sum_s \frac{\hbar\omega_s}{2}\left\{a_s\, a_s^{\dagger} + a_s^{\dagger}\, a_s\right\}$$

$$\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right) = \sum_s \sqrt{\frac{\hbar}{2\,\omega_s\,\varepsilon_0}}\left\{a_s(t)\,\vec{\mathbf{u}}_s\left(\vec{\mathbf{r}}\right) + a_s^{\dagger}(t)\,\vec{\mathbf{u}}^*_{s'}\right\}$$

$$\vec{\mathbf{E}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) = \sum_s \sqrt{\frac{\hbar\omega_s}{2\varepsilon_0}}\left\{a_s(t)\,\vec{\mathbf{u}}_s\left(\vec{\mathbf{r}}\right) - a_s^{\dagger}(t)\,\vec{\mathbf{u}}^*_{s'}\right\} \qquad \text{[II-21]}$$

$$\vec{\mathbf{H}}_{\mathrm{T}}\left(\vec{\mathbf{r}},t\right) = c\sum_s \sqrt{\frac{\hbar}{2\,\omega_s\,\mu_0}}\left\{a_s(t)\,\vec{\nabla}\times\vec{\mathbf{u}}_s\left(\vec{\mathbf{r}}\right) + a_s^{\dagger}(t)\,\vec{\nabla}\times\vec{\mathbf{u}}^*_{s'}\right\}$$

<u>**The Plane Wave Expansion of the Electromagnetic Field Hamiltonian:**</u>

To be definite, we may write an explicit plane wave representation for the field as

$$\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right) = \sum_s\sum_{\sigma=1}^{2} \sqrt{\frac{\hbar}{2\omega_s\,\varepsilon_0\,V}}\;\hat{\mathbf{e}}_{i\sigma}\left\{a_{s\sigma}\exp\left[i\left(\vec{\mathbf{k}}_s\cdot\vec{\mathbf{r}}-\omega_s\,t\right)\right] + a^{\dagger}_{s\sigma}\exp\left[-i\left(\vec{\mathbf{k}}_s\cdot\vec{\mathbf{r}}-\omega_s\,t\right)\right]\right\} \qquad \text{[II-22a]}$$

or[10]

$$\vec{\mathbf{A}}\left(\vec{\mathbf{r}},t\right)=\sum_{s}\sum_{\sigma=1}^{2}\sqrt{\frac{\hbar}{2\omega_{s}\,\varepsilon_{0}}}\left\{a_{s\sigma}(t)\,\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)+a_{s\sigma}^{\dagger}(t)\,\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)\right\} \qquad \text{[II-22b]}$$

where

$$\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)=\frac{1}{\sqrt{V}}\,\hat{\mathbf{e}}_{s\sigma}\,\exp\left[i\,\vec{\mathbf{k}}_{s}\;\vec{\mathbf{r}}\right] \quad \text{and} \quad \vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)=\frac{1}{\sqrt{V}}\,\hat{\mathbf{e}}_{s\sigma}\,\exp\left[-i\,\vec{\mathbf{k}}_{s}\;\vec{\mathbf{r}}\right] \qquad \text{[II-23a]}$$

$$a_{s\sigma}(t)=a_{s\sigma}(0)\,\exp\left[-i\,\omega_{s}\,t\right] \quad \text{and} \quad a_{s\sigma}^{\dagger}(t)=a_{s\sigma}^{\dagger}(0)\,\exp\left[+i\,\omega_{s}\,t\right] \qquad \text{[II-23b]}$$

Of course, $k_s=\left|\vec{\mathbf{k}}_s\right|=\omega_s/c$ in all of these expansions. Further the electric field expansion is given by

$$\begin{aligned}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)&=i\sum_{s}\sum_{\sigma=1}^{2}\sqrt{\frac{\hbar\,\omega_{s}}{2\,\varepsilon_{0}}}\left\{a_{s\sigma}(t)\,\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)-a_{s\sigma}^{\dagger}(t)\,\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)\right\}\\&=i\sum_{s}\sum_{\sigma=1}^{2}\hat{\mathbf{e}}_{s\sigma}\,\mathcal{E}_{s}\left\{a_{s\sigma}\,\exp\left[i\,\vec{\mathbf{k}}_{s}\;\vec{\mathbf{r}}-i\,\omega_{s}\,t\right]-a_{s\sigma}^{\dagger}(t)\,\exp\left[i\,\vec{\mathbf{k}}_{s}\;\vec{\mathbf{r}}-i\,\omega_{s}\,t\right]\right\}\end{aligned} \qquad \text{[II-24a]}$$

where $\mathcal{E}_s=\sqrt{\dfrac{\hbar\omega_s}{2\varepsilon_0\,V}}$ and the magnetic field expansion by

$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)=\frac{i}{c\,\mu_{0}}\sum_{s}\sum_{\sigma=1}^{2}\sqrt{\frac{\hbar\omega_{s}}{2\varepsilon_{0}}}\left\{a_{s\sigma}(t)\left[\hat{\mathbf{k}}_{s}\times\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)\right]-a_{s\sigma}^{\dagger}(t)\left[\hat{\mathbf{k}}_{s}\times\vec{\mathbf{u}}_{s\sigma}\left(\vec{\mathbf{r}}\right)\right]\right\} \qquad \text{[II-24b]}$$

---

10 This expansion is often written as $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\vec{\mathbf{E}}^{(+)}\left(\vec{\mathbf{r}},t\right)+\vec{\mathbf{E}}^{(-)}\left(\vec{\mathbf{r}},t\right)$ where

$$\vec{\mathbf{E}}^{(+)}\left(\vec{\mathbf{r}},t\right)=i\sum_{s}\sum_{\sigma=1}^{2}\hat{\mathbf{e}}_{s\sigma}\,\mathcal{E}_{s}\,a_{s\sigma}\,\exp\left[i\,\vec{\mathbf{k}}_{s}\;\vec{\mathbf{r}}-i\,\omega_{s}\,t\right] \quad \text{and} \quad \vec{\mathbf{E}}^{(-)}\left(\vec{\mathbf{r}},t\right)=-i\sum_{s}\sum_{\sigma=1}^{2}\hat{\mathbf{e}}_{s\sigma}\,\mathcal{E}_{s}\,a_{s\sigma}^{\dagger}(t)\,\exp\left[i\,\vec{\mathbf{k}}_{s}\;\vec{\mathbf{r}}-i\,\omega_{s}\right.$$

In light of Equations [II-18] and [II-21], the Hamiltonian of the radiation field is

$$\mathcal{H}_{rad} = \frac{1}{2}\sum_{\{l\}}\sum_{\sigma=1}^{2}\hbar\,\omega_{\{l\}}\left\{ a_{\{l\}\,\sigma}\, a^{\dagger}_{\{l\}\,\sigma} + a^{\dagger}_{\{l\}\,\sigma}\, a_{\{l\}\,\sigma} \right\} = \sum_{\{l\}}\sum_{\sigma=1}^{2}\hbar\,\omega_{\{l\}}\left\{ \mathcal{N}_{\{l\}\,\sigma} + 1/2 \right\} \qquad [\text{ II-25 }]$$

where

$$\begin{aligned} \sum_{\{l\}} &\equiv \sum_{l_x=-\infty}^{+\infty}\ \sum_{l_y=-\infty}^{+\infty}\ \sum_{l_z=-\infty}^{+\infty} \\ \vec{\mathbf{k}}_{\{\}} &\equiv \frac{2\pi}{L}\left[l_x\,\hat{\mathbf{x}} + l_y\,\hat{\mathbf{y}} + l_z\,\hat{\mathbf{z}} \right] \\ \omega_{\{\}} &\equiv c \left| \vec{\mathbf{k}}_{\{l\}} \right| \end{aligned} \qquad [\text{ II-26 }]$$

The electromagnetic momentum (Poynting vector divided by c^2) is given classical by

$$\vec{\mathcal{M}} = \frac{1}{c^2}\iiint\limits_{\text{cavity}} \vec{\mathbf{E}} \times \vec{\mathbf{H}}\; dV \qquad [\text{ II-27a }]$$

and in terms of the second quantization operators it becomes

$$\begin{aligned} \vec{\mathcal{M}} &= \frac{1}{2}\sum_{\{l\}}\sum_{\sigma=1}^{2}\hbar\,\vec{\mathbf{k}}_{\{l\}}\left\{ a_{\{l\}\,\sigma}\, a^{\dagger}_{\{l\}\,\sigma} + a_{\{l\}\,\sigma}\, a^{\dagger}_{\{l\}\,\sigma} \right\} \\ &= \sum_{\{l\}}\sum_{\sigma=1}^{2}\hbar\,\vec{\mathbf{k}}_{\{l\}}\,\mathcal{N} \end{aligned} \qquad [\text{ II-27b }]$$

Thus, the Fock or number states are eigenstates of both the energy and the momentum of the field.